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Univariate and bivariate characteristics of amplified spontaneous emission and coupled-mode optical systems

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Abstract Spontaneous emission from incoherently excited atoms has a negative exponential intensity probability distribution function (IPDF) In an optical amplifier when the gain is non-linear (due to gain saturation), the intensity fluctuation statistics is significantly altered from that of spontaneous emission as is evidenced by experimental results We have derived the expressions for (i) univariate cumulants of intensity fluctuations up to tenth order from a single mode IPDF and (ii) bivariate moments of intensity fluctuations up to fifth order starting from a bivariate IPDF The IPDFs are negative polynomials (whose powers depend on the number of interactive modes), which rest on the assumption that the variation of intensity has a maximum entropy distribution with the constraint that the total intensity is constant This approximation holds for a heavily saturated source of amplified spontaneous emission (ASE) and some other coupled-mode optical systems Both the normalized cumulants and bivariate moments have shown unique features that depend only on the number of interacting modes in the system The theoretical results have been compared with the experimental ones for ASE The comparison is useful to test the model and to identify the number of interacting modes in coupled-mode optical systems

Keywords Amplified spontaneous emission, coupled-mode optical systems, univariate cumulants and bivariate moments

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1. Introduction

Spontaneously emitted radiation (called spontaneous emission) such as from a laboratory gas discharge tube, arises from incoherently excited group of atoms and it is a random stochastic process. As an addition of randomly phased contributions from different atoms, spontaneous emission has a Gaussian probability distribution for the electric field amplitude [1,2]. This Gaussian electric field probability distribution gives rise to a negative exponential probability distribution function in its intensity fluctuations for the spontaneously emitted light given by [1,2]

$$p_1(I) = \left(\frac{1}{\langle I\rangle}\right) e^{\left(-\frac{I}{\langle I\rangle}\right)}, \tag{1}$$

[where $\langle I\rangle$ is the average intensity] similar to thermal radiation.

Intensity fluctuations in optical systems are often characterized by statistical quantities such as univariate and bivariate moments, and cumulants. Spontaneous emission has the following characteristics [1,2] as found from the distribution expressed by eq. (1) :

(i) First order un-normalized moment (which is same as the first order un-normalized cumulant) $M_1 = K_1 = \langle I\rangle$.

(ii) p-th-order moment (normalized to the appropriate power of first moment), $m_p = p!$;

(iii) p-th-order cumulant (normalized to the appropriate power of first order cumulant), $k_p = (p-1)!$.

In an amplified spontaneous emission (ASE), such as the output from a thin, long, mirrorless, gas laser amplifier, the only source of radiation is the spontaneously emitted radiation at one end of the amplifier. These sources emit radiation that shows a high degree of directionality and coherence [2–18]. The spontaneously emitted radiation generated at one end of the amplifier gets amplified as it travels along the axis of the tube because of the stimulated emissions from the excited atoms. Initially, when the gain is linear (unsaturated region),

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the average intensity increases almost exponentially. When the medium is unsaturated, the output of ASE has negative exponential intensity statistics [2] of spontaneous emission This is so because linear amplification causes a linear change in the scale of intensity and as a result, although the absolute values of the moments change, the normalized moments or cumulants of the distribution stay the same as before. The gain becomes non-linear when the intensity reaches a level at which stimulated emission reduces the number of excited atoms. The average intensity output tends to attain a constant value as the medium is heavily saturated [2–10]. The intensity of the spontaneously emitted field in any mode, however, displays rapid and significant fluctuations about the average intensity even under heavy gain saturation due to the interference effects between the emissions from the incoherently excited sources. The statistical properties of the emitted radiation are significantly changed when appreciable portions of the intensity fluctuations approach this characteristic saturation intensity.

For unsaturated media, each mode could be considered independently. The gain saturation couples the independent spectral, spatial and polarization modes propagating in both directions with a common atomic population [3,4,7,8–10]. Previous experimental measurements had shown unambiguously that there was a significant reduction of the intensity fluctuations of each mode of an ASE source as the gain saturated. Higher order cumulants were found to fall more sharply than the lower order cumulants [2,6,9] as the medium became saturated. Experimental results also had shown that the fluctuations of two orthogonally polarized components of the output from one end of an ASE source were significantly anti-correlated when the medium was heavily saturated and were uncorrelated when the amplifier was unsaturated [8].

The multimode optical system also includes multimode lasers [14–16] where the modes are driven stochastically by spontaneous emission, deterministically by coupled mode dynamics or by a combination of both. When the modal intensities are strong, it is found that the gain is heavily saturated and the total intensity is fairly a constant although the intensities of each mode fluctuate very rapidly and significantly. The systems for which the total intensity is nearly constant include broadband and multimode ASE, non-resonant feedback lasers [17,18] and some cases of multimode lasers. In all these cases the total intensity is stabilized by the common interaction of the modes with the gain of the medium and the resulting mutual cross saturation arising from the saturation of the total energy extracted from the medium.

Our research group [9] derived the intensity probability distribution function (IPDF) for a system of n coupled-modes with the assumptions that the total intensity is constant and the variations in the intensity have a maximum intensity distribution. The assumption holds for heavily saturated coupled-mode optical systems including ASE from a thin, long, mirrorless, gas amplifier The objective of this article is to provide the general expressions for the higher order univariate cumulants and bivariate moments computed from these IPDFs that are described below. The theoretical results of this computation are applicable to the coupled-mode optical systems such as ASE to compare with the experimental results of the univariate and bivariate statistics of the intensity fluctuations. The comparison is useful to test the IPDFs and identify the number of interacting modes in experimental systems when the medium is heavily saturated.

2. Intensity probability distribution functions

The result for the single mode IPDF that is applicable for heavily saturated coupled-mode optical systems as found by our research group [9], is a negative polynomial distribution whose power depends on the number of interacting modes (n). The single mode IPDF is given by

$$p_n(I_1) = (n-1)I^{1-n}(I-I_1)^{n-2}, \tag{2}$$

where I is the constant total intensity given by $I = \sum_{i=1}^{n} I_i$, I_i being the intensity of the i-th mode. This result is same as was obtained by Masalov [1] who derived the result by renormalizing the total intensity of multimode Gaussian radiation to have a constant value. In the limit of large number of modes n, the preceding polynomial distribution reduces to the negative exponential distribution [eq. (1)] characteristic of spontaneous emission and thermal radiation [9]. The bivariate IPDF $p_n(I_1,I_2)$ for the modes 1 and 2 as found by our research group [9] is

$$p_n(I_1, I_2) = (n-1)(n-2)I^{1-n}(I-I_1-I_2)^{n-3}. \tag{3}$$

Eq. (3) when integrated for all possible values of I_2, reduces to eq. (1), the single mode IPDF expressed by eq. (1), as expected.

In the following section, we show the results for the general expressions for the univariate cumulants of up to the tenth order obtained from the single mode IPDF given by eq. (2) and the general expressions for the bivariate moments from the bivariate IPDF given by eq. (3). Knowledge of higher order cumulants is important since experiments had shown more rapid fractional reduction of the higher order cumulants in comparison with the lower order cumulants as function of gain saturation in the medium for ASE [2,6,9].

3. Univariate cumulants and bivariate moments of the intensity fluctuations

In the computation of statistics of a signal, cumulants are always preferred over moments. This is because the n-th-order cumulant represents the 'true' n-th-order moment of a signal where the effects of all lower order moments are washed out, in contrast to the n-th-order moment which has dependence, in part, of all lower order moments. There is another advantage of computing cumulants (compared to moments) for experimental signals. The true cumulant of an experimental signal only can be obtained by subtracting the cumulant of the noise present in the signal (obtained by blocking the signal) from the cumulant values of the signal with noise (assuming noise is independent of the signal) [2,19,20]. However, if the noise is not independent of the signal, the above statement does not apply.

Stuart and Ord [21] provided the relationship between the univariate moments and univariate cumulants of a distribution. We have used those relations to determine the univariate cumulant values up to the tenth order for the IPDF expressed by eq. (2). We used MAPLE release V for all computations of un-normalized cumulants (K_p) and for simplification of the results into the forms as shown below :

$$M_p = \langle I_1^p \rangle = \frac{I^p p!(n-1)!}{(n+p-1)!},$$

$$K_1 = \frac{I}{n},$$

$$K_2 = \frac{I^2(n-1)}{n^2(n+1)},$$

$$K_3 = \frac{2I^3(n-1)(n-2)}{n^3(n+1)(n+2)},$$

$$K_4 = \frac{6I^4(n-1)(n^3-4n^2-n+6)}{n^4(n+1)^2(n+2)(n+3)},$$

$$K_5 = \frac{24I^5(n-1)(n-2)(n^3-6n^2-5n+12)}{n^5(n+1)^2(n+2)(n+3)(n+4)},$$

$$K_6 = 120\frac{I^6(n-1)(n^7-10n^6-10n^5+108n^4}{n^6(n+1)^3(n+2)^2(n+3)(n+4)(n+5)}$$

$$\frac{+55n^3-320n^2-52n+240)}{n^6(n+1)^3(n+2)^2(n+3)(n+4)(n+5)},$$

$$K_7 = 720\frac{I^7(n-1)(n-2)(n^7-14n^6-18n^5+228n^4}{n^7(n+1)^3(n+2)^2(n+3)(n+4)(n+5)(n+6)}$$

$$\frac{+279n^3-788n^2-396n+720)}{n^7(n+1)^3(n+2)^2(n+3)(n+4)(n+5)(n+6)},$$

$$K_8 =$$

$$5040\frac{I^8(n-1)(n^{11}-19n^{10}-37n^9+669n^8+1195n^7-6963n^6}{n^8(n+1)^4(n+2)^2(n+3)^2(n+4)(n+5)(n+6)(n+7)}$$

$$\frac{-10629n^5+30421n^4+26150n^3-53580n^2-17352n+30240)}{n^8(n+1)^4(n+2)^2(n+3)^2(n+4)(n+5)(n+6)(n+7)}$$

$$K_9 = 40320\frac{I^9(n-1)(n-2)(n^{12}-23n^{11}-97n^{10}+1169n^9+5703n^8-10203n^7}{n^9(n+1)^4(n+2)^3(n+3)^2(n+4)(n+5)(n+6)(n+7)(n+8)}$$

$$\frac{-77245n^6-12643n^5+290974n^4+115300n^3-445992n^2-108576n+241920)}{n^9(n+1)^4(n+2)^3(n+3)^2(n+4)(n+5)(n+6)(n+7)(n+8)}$$

$K_{10} = 362880\ I^{10}\ (n-1)\ (n^{16}-29n^{15} - 160n^{14}+2472n^{13}+15736n^{12}-41106n^{11}-412848n^{10} - 23752n^9+3949531n^8+2746383n^7-17318788n^6 - 12850608n^5+38866592n^4+20283792n^3-42395904n^2\ 10236672n+17418240)/[n^{10}(n+1)^5(n+2)^3(n+3)^2(n+4)^2(n+5)(n+6)(n+7)(n+8)(n+9)]$.

Dividing the univariate cumulants by appropriate power of the first order cumulant (K_1) of the signal, we find clearly that the normalized univariate cumulants ($k_p = K_p/K_1{}^p$) have unique features that depend only on the number of interacting modes (n). It is evident that the asymptotic values ($n \to \infty$) of the normalized univariate cumulants k_p are :

$k_1 = 1$, $k_2 = 1$, $k_3 = 2$, $k_4 = 6$, $k_5 = 24$, $k_6 = 120$, $k_7 = 720$, $k_8 = 5040$, $k_9 = 40320$, $k_{10} = 362880$.

These values are identical with the values expected for spontaneous emission, unsaturated amplified spontaneous emission, and thermal radiation [$k_p = (p-1)!$] characteristic of the distribution expressed by eq. (1). This shows that spontaneous emission with negative exponential probability distribution applies to a mode with heavily saturated gain interacting with a very large number of other modes.

We computed the numerical values of the normalized comulants as fraction of their values for thermal light (that is, $\kappa_p = k_p/(p-1)!$ from the above expressions. For example,

$$\kappa_5 = \frac{(n-1)(n-2)(n^3-6n^2-5n+12)}{(n+1)^2(n+2)(n+3)(n+4)}.$$

From the computations, we find that all these cumulants (κ_p) decrease from their initial value of 1 (for $n = \infty$) and tend towards zero (except κ_2 which becomes 0.333) as n goes down to 2. The higher order κ values decrease more rapidly than the lower order κ values as n decreases. For example, using the expressions for κ_p, we found that for n = 50, 20, 10, 4 and 2 the values of κ_2 = 0.961, 0.905, 0.808, 0.600 and 0.333 respectively and the values of κ_3 are 0.887, 0.740, 0.545, 0.200 and 0.000 respectively. For the same values of n, the values of κ_8 are 0.300, 0.009, –0.025, 0.002 and 0.000 respectively and the values of κ_{10} are 0.118, –0.030, –0.002, 0.000 and 0.000 respectively.

One can determine experimentally the normalized cumulant values of the intensity fluctuations for an experimental signal by computing the moments of the distribution from its intensity time series. However, in many experimental situations the average value (the first moment) of the signal is not meaningful or cannot be determined absolutely. The examples include the experiments where ac-coupling of the signals and the presence of amplifier noise of the detection system make it difficult to determine the absolute average and absolute zero of the signals. In these situations one can compute the normalized univariate cumulant $k'_p\left[= K_p/K_2^{p/2}\right]$ of the signal normalized to the appropriate power of the second order univariate cumulant (K_2) of the signal [6 9] From the preceding expressions of un-normalized cumulants, we determined k'_p values to compare with the available experimental results [2,6,9] of the k'_3, k'_4, k'_5 values for a selected linearly polarized ASE output of 3.51 μm radiation from a thin, long, mirrorless, gas amplifier containing Xe-He. At the longest length of 3 meters of the amplifier and under most heavily saturated condition, the experimental values of k'_3, k'_4 and k'_5 decreased from their unsaturated values (which corresponds to $n \to \infty$) of 2, 6 and 24 to about 1.4, 2.4 and 4, respectively. We found by comparison with the computed values that these experimental values correspond to approximately $n = 10$ for the experimental system. Experimentally, higher order cumulants decrease more rapidly than the lower order cumulants with the increasing length of the amplifier (that is, with increasing saturation of the medium). This is clearly predicted from this computation, which shows that as n decreases from infinity (corresponding to unsaturated radiation) higher order cumulants fall more rapidly than the lower order cumulants. To our knowledge, no experiment has been done under more heavily saturated conditions for ASE.

We have also derived the expressions for the un-normalized bivariate moments of up to the fifth order using the bivariate IPDF as given by eq. (3). As is evident from eq. (3), the general expressions for the evaluation of bivariate moments involve double integrals. We first evaluated the integrals using MAPLE release V for different individual integer values of n such as 3, 4, 5 *etc*. Then we generalized the results for all possible positive integer values of n ($n > 2$) by fitting the numerical results. The latter involves solution of several linear equations (the number of equations depends on the order of the bivariate moment) which we solved also using MAPLE. The denominator of each bivariate moment was found to be a polynomial expression whose coefficients were exact integers. Following are the results we found for the un-normalized bivariate moments of up to the fifth order :

M_1 = first univariate moment $= \dfrac{I}{n}$,

CM_2 = second univariate central moment

$$= \left\langle (I - \langle I \rangle)^2 \right\rangle = \frac{I^2(n-1)}{n^2(n+1)},$$

$$M_{11} = \frac{I^2}{n(n+1)},$$

$$M_{21} = \frac{2I^3}{n(n+1)(n+2)},$$

$$M_{31} = \frac{6I^4}{n(n+1)(n+2)(n+3)},$$

$$M_{22} = \frac{4I^4}{n(n+1)(n+2)(n+3)},$$

$$M_{41} = \frac{24I^5}{n(n+1)(n+2)(n+3)(n+4)},$$

$$M_{32} = \frac{12I^5}{n(n+1)(n+2)(n+3)(n+4)}.$$

It is clear from the above expression that the normalized bivariate moments $m_{pq} [= M_{pq} / (M_1)^{p+q}]$ have unique values that depend on the number of interactive modes n. The asymptotic values (for $n \to \infty$) of m_{21}, m_{31}, m_{22}, m_{41}, m_{32} are 2, 6, 4, 24 and 12 respectively. Experimentally, normalized bivariate moments of intensity fluctuations can be determined for heavily saturated optical systems such as between the two orthogonally polarized components of the output from an ASE source. Comparison of the computed normalized bivariate moments with the experimentally determined values will test the validity of the bivariate IPDF and identify the number of interacting modes in heavily saturated optical systems. In experimental situations where average value of the signal cannot be determined, one can compute bivariate moment of the signal normalized to the appropriate power of the second central moment (CM_2) of one of the signals.

If the modes I_1 and I_2 are independent, then $p_n(I_1,I_2) = P_n(I_1) \cdot p_n(I_2)$. As a result, $M_{pq} = M_p \cdot M_q$. By direct multiplication of the univariate moments, we get

$$M_2 \cdot M_1 = \frac{2I^3}{n^2(n+1)},$$

$$M_3 \cdot M_1 = \frac{6I^4}{n^2(n+1)(n+2)},$$

$$M_2 \cdot M_2 = \frac{4I^4}{n^2(n+1)^2},$$

$$M_4 \cdot M_1 = \frac{24I^5}{n^2(n+1)(n+2)(n+3)},$$

$$M_3 \cdot M_2 = \frac{12I^5}{n^2(n+1)^2(n+2)},$$

It is obvious that when n is very large, the values obtained from the preceding expressions for M_{pq} will be very close to those from the above expressions of $M_p \cdot M_q$, as expected.

4. Conclusion

We have reported the general expressions for the univariate cumulants up to the tenth order from a single mode probability distribution and bivariate moments up to the fifth order from a bivariate probability distribution for the intensity fluctuations. The probability distribution functions used in the computations are negative polynomial expressions, which are based on the assumptions that the total intensity is constant and the variations in the intensity have a maximum entropy distribution. These assumptions hold in heavily saturated coupled-mode optical systems including ASE from a thin, long, mirrorless, gas laser amplifier. The normalized univariate cumulants and bivarate moments both have shown characteristic features that depend only on the number of interactive modes in the system. Knowledge of higher order cumulants is important since a cumulant of a particular order represents the 'true moment' of that order and experimentally it was found that higher order cumulants values decreased more rapidly in an ASE as the medium gets heavily saturated. The asymptotic values of the univariate cumulants of all the orders agree with the cumulant values from a negative exponential probability distribution function (such as for spontaneous emission, thermal radiation and unsaturated ASE), as expected. The values of the normalized cumulants, when calculated as fraction of their thermal values, are all found to decrease from unity as the number of interacting modes decreased. The computed values of the higher order cumulants decreased more rapidly than lower order cumulants. Previous experimental results for ASE also showed that the univariate cumulants value decreased with increasing saturation of the medium with higher order cumulants decreasing more rapidly than lower order cumulants. Asymptotic values of the bivariate moments agree with the values as expected from the product of the corresponding univariate moments. Comparison was made with the computed values of cumulants with the available experimental data to determine the number of interacting modes in ASE. Comparişon of theoretical results from this computation with the experimental results is useful to test the validity of the intensity probability distribution functions and identify the number of interacting modes in coupled-mode experimental optical systems including ASE from a thin, long, mirrorles, gas laser amplifier.

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